Real Analysis-I

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1 Mathematical Logic

1.1 Introduction: The meaning of "Logic" is the "Science of reasoning". The language of mathematics is very neat and concise and surpasses every other language in its precision. The study of logic through the use of mathematical symbols is called **mathematical logic.** It is also called **"Symbolic logic"**. Since symbols are abstract and neutral, they give clear expression to our thoughts. The mathematical approach to logic was first propounded by British mathematician **George Bool.** On this account, the mathematical logic is also called **Boolean logic.**

Mathematical logic deals with the topic like statement, negation, connectives, compound statements, conjunction, disjunction, duality, truth table, conditional and bi-conditional statements, valid arguments, tautology etc.

1.2 Statement: An sentence is called a statement if it is either true or false but not both.

1.3 Statement Letters: The symbols which are taken to represent the statement, are called statement letters. Statements are denoted by letters p, q, r,.....

For examples consider the following sentences:

1. Delhi is the Capital of India.
2. 5 is less than 2.
3. Where are you going?
4. Today is Monday.
5. In plane geometry sum of the angles of a triangle is two right angles.

In the above sentences (1), (2) and (5) are statements but (3) and (4) are not statements because (1) and (5) are true and (2) is false while (3) and (4) neither true nor false.

1.4 Connectives: Any two statements may be combined in different ways to form new statement. To form new statement, the words, which are used are called **connectives.** We shall now discuss three most basic and fundamental connectives, which are negation, conjunction and disjunction. These are associated with English words 'Not', 'And', 'Or' respectively. Some other connectives will be discussed subsequently. These connectives are given below:

English words	Name of connective	Symbol	order
Not	Negation	$\sim$ or $\neg$	1
And	Conjunction	$\wedge$	2
Or	Disjunction	$\vee$	3
One way implication	Conditional	$\Rightarrow$	4
If and only if (Two way implication)	Bi-conditional	$\Leftrightarrow$	5

1.5 Negation: If p is any statement, then the denial of statement p is called the **negation** of statement p and is written as $\sim$ p or $\neg$ p

The negation of statement p is formed by inserting the word 'not' in p.

Examples:

(i) Let p be the statement: 6 is a factor of 18, then negation of p is
$\sim$ p = 6 is not a factor of 18.

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(ii) Let q: Jaipur is in Bihar.
$\sim q$ = Jaipur is not in Bihar.

1.6 Conjunction: If two statements are combined by using the logical connective "and" then resulting statement is called a **conjunction.** The conjunction of statements p and q is denoted by $p \wedge q$.

For example, let

p : Monson is very good this year.

q : The rivers Ganga and Yamuna are rising.

then

$p \wedge q$: Monsoon is very good this year and the rivers Ganga and Yamuna are rising.

1.7 Disjunction: If two statements are combined by using the logical connective "or", then resulting statement is called **disjunction.**

The Disjunction of two statements p and q is denoted by $p \vee q$.

For example, let

$p : 6 \leq 8$

and q : 4 is integer, then

$p \vee q$: $6 \leq 8$ or 4 is an integer.

1.8 Conditional: If two statement are combined by using the logical connective "if .....then', then the resulting statement is a called a **conditional statement.**

The conditional statement of two statements p and q (in this order) is denoted by $p \Rightarrow q$. Here p is called **antecedent** and q is called **consequent.**

For example, let

p :2 + 5 = 7 and

q : 9 is an integer, then

$p \Rightarrow q$: 'if 2 + 5 = 7, then 9 is an integer'.

Note: The statement $p \Rightarrow q$ has four possibilities depends on p and q.	
Case I :	If p is true, q is false, then $p \Rightarrow q$ is false.
Case II :	If p is true, q is true, then $p \Rightarrow q$ is true.
Case III :	If p is false, q is true, then $p \Rightarrow q$ is true.
Case IV :	If p is false, q is false, then $p \Rightarrow q$ is true.

1.9 Bi-conditional: If two statement are combined by using the logical connective "if and only if", then the resulting statement is a called a **bi-conditional statement.**

The Bi-conditional statement of two statements p and q is denoted by $p \Leftrightarrow q$.

For example, let

p : 3 divides 9 and q : divides 12, then

$p \Leftrightarrow q$: '3 divides 9 if and only if 4 divides 12'.

1.10 Joint Denial: The word "NOR" is combination of NOT and OR where NOT and OR means negation and disjunction respectively. Let p and q be two statements then "$p \downarrow q$" is called joint denial or "NOR" statement and we read it as "Neither p nor q". This statement "$p \downarrow q$" can also be written as "$p \downarrow q$" $\Leftrightarrow \sim (p \vee q)$.

"$p \downarrow q$" statement is true only when p and q both are false.

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1.11 NAND Statement: The word "NAND" is a combination of NOT and AND where NOT and AND means negation and conjunction respectively. The statement "p↑q" is called "NAND" statement and can be written as $p \uparrow q \Leftrightarrow \sim (p \wedge q)$.

This statement "p↑q" is false only if p and q both are true.

1.12 Types of Conditional Statement: Let p and q be two statements, then there are three other Condition statements related to $p \Rightarrow q$.

(i) **Converse Implication:** The statement $q \Rightarrow p$ is called converse implication of the statement $p \Rightarrow q$.

(ii) **Inverse Implication:** The statement $\sim p \Rightarrow \sim q$ is called inverse implication of the statement $p \Rightarrow q$.

(iii) **Contrapositive Implication:** The statement $\sim q \Rightarrow \sim p$ is called contra positive implication of the statement $p \Rightarrow q$.

1.13 Use of Brackets: The use of brackets in the statement has an important role. By using brackets in the statements, the meaning or explanation of the statements are completely different. There are some rules related to the brackets.

(i) If the negation (~) is repeated in the statement then there is no need of bracket.

Exp.: $\sim(\sim p)$ and $\left(\sim\left(\sim p\right)\right)$ both have same meaning.

(ii) If the connectives of same type are present in a statement, then brackets are applied from left.

Exp.: $p \vee q \vee r \vee s = \left[\left\{\left(p \vee q\right) \vee r\right\} \vee s\right]$

(iii) If the different connectives are used in the statement, then we remove the bracket of lower order connective, but we can not remove the bracket of higher order connective.

Exp.: $p \Rightarrow (q \wedge r) = p \Rightarrow q \wedge r$

Here order of $\wedge$ is less than the order of $\Rightarrow$, hence bracket is removed.

1.14 Types of Sentences: There are two types of sentences.

(i) **Simple Sentences (Statement):** A statement which has no connectives is called simple statement or Atomic statement.

Exp.: He is a boy.

(ii) **Compound Sentences:** A statement which is formed by two simple statements through connective is called compound statement or molecular statement (sentence), Some compound statements formed by using p and q are:

Exp.: $\sim p$, $p \vee q$, $(p \wedge q) \vee (\sim p)$, $p \wedge (\sim q)$.

1.15 Statement Formulas: The compound statement which is obtained from statement variables, using finite number of connectives is called **statement formulas.**

Exp.: $p \wedge q$, $p \vee q$, $p \wedge (\sim q)$ etc.

1.16 Truth Values and Truth Table: Since every statement has a unique value. This unique value is called **Truth Value.** These truth values are "True" or "False". Hence the truth value of every Truth function is obtained by the truth values of its components. The range of truth function in {T, F}. There will be 2^n

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truth function having n statements. T and F represent the truth value, True and False respectively. These truth values are also represented by 1 and 0.

A table indicating the truth values of one or more statement is called a **truth table.**

The truth tables for one statement "p", two statements "p, q", three statements "p, q, r" are shown below in figure (i), (ii) and (iii) respectively.

p	OR	p
T		1
F		0

(i)

p	q	OR	p	q	OR	p	q	r	OR	p	q	r
T	T		1	1		T	T	T		1	1	1
T	F		1	0		T	T	F		1	1	0
F	T		0	1		T	F	T		1	0	1
F	F		0	0		F	T	T		0	1	1
						T	F	F		1	0	0
						F	T	F		0	1	0
						F	F	T		0	0	1
						F	F	F		0	0	0

(ii) (iii)

1.17 Truth Table for Negation: Let p be any statement. The truth value of p and ~ p are always opposite to each other.

p	~ p
T	F
F	T

OR

p	~ p
1	0
0	1

In the above table, the first line states that if p is true then ~ p is false and the second line states that if p is false then ~ p is true.

1.18 Truth Table for Conjunction: Let p and q be two statements. Then conjunction $p \wedge q$ is defined to be true when p and q both are true. The truth table of $p \wedge q$ is given below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

OR

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

1.19 Truth Table for Disjunction: Let p and q be two statements then disjunction $p \vee q$ has the truth value false only if both are false. The truth table is given below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

OR

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

1.20 Truth Table for Conditional: Let p and q be two simple statements, then $p \Rightarrow q$ is called the conditional statement. This statement has the true value false only if antecedent p is true and consequent q is false. The truth table is given below:

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p	q	p ⇒ q
T	T	T
T	F	F
F	T	T
F	F	T

OR

p	q	p ⇒ q
1	1	1
1	0	0
0	1	1
0	0	1

1.21 Truth Table for Bi-conditional: Let p and q be any two simple statements. Then the statement p ⇔ is called **Bi-conditional.** This statement can be written as $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$. The truth table is given below:

p	q	p ⇔ q
T	T	T
T	F	F
F	T	F
F	F	T

OR

p	q	p ⇔ q
1	1	1
1	0	0
0	1	0
0	0	1

1.22 Truth Table for Joint Denial: Let p and q be two statements. The statement p↓q is called **JOINT DENIAL,** which is read as "Neither p nor q". p↓q has truth value 'True' when both p and both are false. The truth table is given below:

p	q	p↓q
T	T	F
T	F	F
F	T	F
F	F	T

OR

p	q	p↓q
1	1	0
1	0	0
0	1	0
0	0	1

1.23 Truth Table for NAND Statement: Let p and q be two statements. The statement p↑q is called NAND statement. p↑q is false only when both p and q are true. The truth table is given below:

p	q	p↑q
T	T	F
T	F	T
F	T	T
F	F	T

OR

p	q	p↑q
1	1	0
1	0	1
0	1	1
0	0	1

Working Rules For Solving Problems:

Rule I: A truth table indicates the truth value of number of statements and their compound statements in a compact form.

Rule II: If there are n statements, then there are 2^n rows in the truth table.

Rule III: The negation ~ p of the statement p is the denial of p.

Rule IV: The conjunction of statements p and q is denoted by $p \wedge q$ and is true only when p and q both are true.

Rule V: The disjunction of statements p and q is denoted by $p \vee q$ and true if at least one of p and q is true.

Rule VI: The conditional statements p and q is denoted by $p \Rightarrow q$ and is true except when p is true and q is false.

Rule VII: The Bi-conditional statement of statements p and q is denoted by $p \Leftrightarrow q$ and is true only when p and q have same truth value.

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Solved Examples

Example 1: Which of the following sentences are statements:

(i) $1 + 2 + 3 = 1 \times 2 \times 3$ (ii) $a \in \{a, b, c\}$

(iii) It may rain today. (iv) 5 is a prime number.

(v) Come here! (vi) He is honest.

Solution:

(i) Since $1 + 2 + 3 = 1 \times 2 \times 3$ or $6 = 6$, which is true, hence (i) is a statement.

(ii) $a \in \{a, b, c\}$ is true. Hence, it is a statement.

(iii) "It may rain today". It is not a declarative sentence. Hence it is not a statement.

(iv) "5 is a prime number" is a declarative sentence having truth value 'True'. It is a statement.

(v) "Come here". This is not a declarative sentence. Hence it is not a statement.

(vi) "He is honest" is a declarative sentence. Hence it is a statement.

Example 2: Write the truth values of the following statements:

(i) $ax^2 + bx + c = 0$ may have non-real roots.

(ii) The capital of America is New York.

(iii) Intersection of two non-empty sets is always non empty.

(iv) There are only finite number of integers.

Solution:

(i) T (ii) T

(iii) F (iv) F

Example 3: Write the negation of the following statements:

(i) $2 + 5 = 7$ (ii) $7 \leq 13$

(iii) All doctors are men. (iv) Lucknow is in U.P.

Solution:

(i) $2 + 5 \neq 7$ (ii) $7 > 13$

(iii) All doctors are not men. (iv) Lucknow is not in U.P.

Example 4: Find the truth value of the statement $\sim p$ if the statement p is:

(i) $(3 + 9) - 7 = 4$

(ii) $2^{(3^2)} \neq (2^3)^2$

(iii) $7 \cdot 3$ is an irrational number.

Solution:

(i) $p : (3 + 9) - 7 = 4$

$\therefore \sim p : (3 + 9) - 7 \neq 4$

$\because (3 + 9) - 7 = 5$ and $5 \neq 4$

$\therefore \sim p$ is T.

(ii) $p : 2^{(3^2)} \neq (2^3)^2$

$\therefore \sim p : 2^{(3^2)} = (2^3)^2$

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$\because 2^{(3^2)} = 512$ and $\left(2^3\right)^2 = 64$ and $512 \neq 64$

$\therefore \sim p$ is F.

(iii) p : 7·3 is an irrational number.

$\therefore \sim p$: 7·3 is not an irrational number.

$\because 7 \cdot 3 \in Q$,

$\therefore$ The truth value of $\sim p$ is T.

Example 5: Let p and q stand for the statements: 'Ravi is a Carpenter' and 'He is hard working' respectively. Describe the following statements:

(i) $p \wedge q$ (ii) $\sim p \wedge q$

(iii) $p \wedge \sim q$ (iv) $\sim p \wedge \sim q$

Solution: We have p: Ravi is a Carpenter. and q: He is hard working.

(i) $p \wedge q$: Ravi is a Carpenter and he is hard working.

(ii) $\sim p \wedge q$: Ravi is not a Carpenter and he is hard working.

(iii) $p \wedge \sim q$: Ravi is a Carpenter and he is not hard working.

(iv) $\sim p \wedge -q$: Ravi is not a Carpenter and he is not hard working.

Example 6: Find the truth values of the following statements:

(i) 2 divides 6 and 3 + 5 = 8

(ii) 5 divides 10 and 8 + 2 = 12

Solution:

(i) Truth value of "2 divides 6" is T.

Truth value of "3 + 5 = 8" is T.

$\therefore$ Truth value of "2 divides 6 and 3 + 5 = 8" is T.

(i) Truth value of "5 divides 10" is T.

Truth value of "8 + 2 = 12" is F.

$\therefore$ Truth value of "5 divides 10 and 8 + 2 = 12" is F.

Example 7: Construct the truth tables of:

(i) $\sim p \wedge q$ (ii) $p \wedge \sim q$

(iv) $\sim (p \wedge q)$

Solution:

(i) Truth table of $\sim p \wedge q$

p	q	~p	~p ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(ii) Truth table of $p \wedge \sim q$

p	q	~q	p ∧ ~q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

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(iii) **Truth table of $\sim(p \wedge q)$**

p	q	$(p \wedge q)$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Example 8: Let p and q stand for the statement "Rinki is tall" and "Pinki is beautiful" respectively. Describe the following statements:

(i) $p \vee q$ (ii) $\sim p \vee q$

(iii) $p \vee \sim q$ (iv) $\sim p \vee \sim q$

Solution: p: Rinki is tall

q: Pinki is beautiful

(i) $p \vee q$: Rinki is tall or Pinki is beautiful.

(ii) $\sim p \vee q$: Rinki is not tall or Pinki is beautiful.

(iii) $p \vee \sim q$: Rinki is tall or Pinki is not beautiful.

(iv) $\sim p \vee \sim q$: Rinki is not tall or Pinki is not beautiful.

Example 9: Find the truth values of the following compound statements:

(i) Honesty is the best policy or $2 < 5$

(ii) Honesty is the best policy or $3 > 8$

(iii) Honesty is the worst policy or $6 \geq 2$

(iv) Honesty is the worst policy or $9 < 7$

Solution: We know that $p \vee q$ is true only when at least one of the p and q is true.

(i) Truth value of "Honesty is the best policy" is T.
Truth value of $2 < 5$ is T.
$\therefore$ Truth value of "Honesty is the best policy or $2 < 5$" is T.

(ii) Truth value of "Honesty is the best policy" is T.
Truth value of $3 > 8$ is T.
$\therefore$ Truth value of "Honesty is the best policy or $3 < 8$" is T.

(iii) Truth value of "Honesty is the best policy" is T.
Truth value of $6 \geq 2$ is T.
$\therefore$ Truth value of "Honesty is the best policy or $6 \geq 2$" is T.

(iv) Truth value of "Honesty is the best policy" is T.
Truth value of $9 < 7$ is T.
$\therefore$ Truth value of "Honesty is the best policy or $9 < 7$" is T.

Example 10: Construct the truth table of the following statements:

(i) $p \vee \sim q$

(ii) $\sim(p \vee q)$

(iii) $\sim(\sim p \vee q)$

(iv) $\sim(p \vee q) \wedge (\sim p \wedge \sim q)$

(v) $\sim(p \vee \sim q) \wedge (\sim p \vee r)$

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Solution:

(i) Truth table of $p \vee \sim q$:

p	q	~q	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ii) Truth table of $\sim (p \vee q)$:

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(iii) Truth table of $\sim (\sim p \vee q)$:

p	q	~p	$(\sim p \vee q)$	$\sim (\sim p \vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

(iv) Truth table of $\sim (p \vee q) \wedge (\sim p \wedge \sim q)$:

p	q	~p	~q	$(\sim p \vee q)$	$(\sim p \wedge \sim q)$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

(v) Truth table of $\sim (p \vee \sim q) \wedge (\sim p \vee r)$:

p	q	r	~p	~q	$p \vee \sim q$	$(\sim p \wedge \sim q)$	$\sim p \vee r$	$-(p \vee \sim q) \wedge (\sim p \vee r)$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F
T	F	T	F	T	T	F	T	T
F	T	T	T	F	F	T	T	T
T	F	F	F	T	T	F	F	F
F	T	F	T	F	F	T	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Example 11: Let p and q stand for the statements "Allahabad is in U.P." and 2 + 3 = 5 respectively. Describe the following conditional statements:

(i) $p \Rightarrow q$ (ii) $\sim p \Rightarrow q$

(iii) $p \Rightarrow \sim q$ (iv) $\sim p \Rightarrow \sim q$

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Solution: We have, p: Allahabad is in U.P.
and $q = 2 + 3 = 5$

(i) $p \Rightarrow q$: If Allahabad is in U.P. then $2 + 3 = 5$
(ii) $\sim p \Rightarrow q$: If Allahabad is not in U.P. then $2 + 3 = 5$
(iii) $p \Rightarrow \sim q$: If Allahabad is in U.P. then $2 + 3 \neq 5$
(iv) $\sim p \Rightarrow \sim q$: If Allahabad is not in U.P. then $2 + 3 \neq 5$

Example 12: Let p and q stand for the statements "Old is gold" and "Work is worship" respectively, then find the truth value of the following conditional statements:

(i) $p \Rightarrow q$ (ii) $p \Rightarrow \sim q$
(iii) $q \Rightarrow \sim p$ (iv) $\sim q \Rightarrow p$
(iv) $\sim p \Rightarrow q$ (vi) $\sim p \Rightarrow \sim q$

Solution: We have, p: Old is gold
and q: Work is worship

$\therefore$ p and q both are true and $\sim p$ and $\sim q$ both are false.

(i) The truth value of $p \Rightarrow q$ is T
(ii) The truth value of $p \Rightarrow \sim q$ is F
(iii) The truth value of $q \Rightarrow \sim p$ is F
(iv) The truth value of $\sim q \Rightarrow p$ is T
(v) The truth value of $\sim p \Rightarrow q$ is T
(vi) The truth value of $\sim p \Rightarrow \sim q$ is T

Example 13: Construct the truth table for the following statements:

(i) $\sim p \Rightarrow q$
(ii) $\sim p \Rightarrow (q \Rightarrow p)$
(iii) $(p \Rightarrow q) \Rightarrow (p \wedge q)$

Solution:

(i) Truth table of $\sim p \Rightarrow q$

p	q	$\sim p$	$\sim p \Rightarrow q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

(ii) Truth table of $\sim p \Rightarrow (q \Rightarrow p)$

p	q	$\sim p$	$q \Rightarrow p$	$\sim p \Rightarrow (q \Rightarrow p)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

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(iii) Truth table of $(p \Rightarrow q) \Rightarrow (p \wedge q)$

p	q	$p \Rightarrow q$	$(p \wedge q)$	$(p \Rightarrow q) \Rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Example 14: Let p and q stand for the statements "Sita speaks English" and "Geeta speaks Punjabi". Describe the following statements:

(i) $p \Leftrightarrow q$ (ii) $q \Leftrightarrow p$

(iii) $\sim p \Rightarrow q$ (iv) $p \Leftrightarrow \sim q$

(v) $\sim p \Leftrightarrow \sim q$

Solution: We have, p: Sita speaks English

and q: Geeta speaks Punjabi

(i) $p \Leftrightarrow q$: Sita speaks English if Geeta speaks Punjabi.

(ii) $q \Leftrightarrow p$: Geeta speaks Punjabi if Sita speaks English.

(iii) $\sim p \Rightarrow q$: Sita does not speaks English if Geeta speaks Punjabi.

(iv) $p \Leftrightarrow \sim q$: Sita speaks English if Geeta does not speaks Punjabi.

(v) $\sim p \Leftrightarrow \sim q$: Sita does not speaks English if Geeta does not speaks Punjabi.

Example 15: Construct the truth table of the following statements:

(i) $(p \Rightarrow q) \vee \sim (p \Leftrightarrow \sim q)$ (ii) $(p \vee q) \Leftrightarrow r$

Solution:

(i) Truth table of $(p \Rightarrow q) \vee \sim (p \Leftrightarrow \sim q)$

p	q	$\sim q$	$p \Rightarrow q$	$p \Leftrightarrow \sim q$	$\sim (p \Leftrightarrow \sim q)$	$(p \Rightarrow q) \vee (p \Leftrightarrow \sim q)$
T	T	F	T	F	T	T
T	F	T	F	T	F	F
F	T	F	T	T	F	T
F	F	T	T	F	T	T

(ii) Truth table of $(p \vee q) \Leftrightarrow r$

p	q	r	$p \vee q \Leftrightarrow r$
T	T	T	T
T	T	F	F
T	F	T	T
F	T	T	T
T	F	F	F
F	T	F	F
F	F	T	F
F	F	F	T

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Example 16: Write the negation of $(p \vee q) \wedge (p \Rightarrow q)$ *(A.U. 2012)*

Solution:

(i) Truth table of $\sim [(p \vee q) \wedge (p \Rightarrow q)]$

p	q	$p \vee q$	$p \Rightarrow q$	$(p \vee q) \wedge (p \Rightarrow q)$	$\sim [(p \vee q) \wedge (p \Rightarrow q)]$
T	T	T	T	T	F
T	F	T	F	F	T
F	T	T	T	T	F
T	F	F	T	F	T

Exercise 1.1

1. Which of the following are statement:

(i) Grass is green.

(ii) Is the number 7 a prime ?

(iii) Red roses are beautiful.

(iv) $x^2 + y^2 \geq 0$

(v) $\{2x + 3y = 5 : (x, y) = (1, 1), x, y \in N\}$

2. Write the negation of the following are statement:

(i) The cube of 3 is 27. (ii) 5 divides 14.

(iii) Lucknow is in Bihar. (iv) Product of 2 and 6 is 22.

(v) $8 > 5$

3. Find the truth value of the statement $\sim p$ if the statement p is:

(i) $3 + 4 = 7$ (ii) $\log_2 8 = 4$

(iii) $2 \times 5 = 12$ (iv) $5 - 3 = 2$

4. Let p and q stands for the statement: "2 + 3 = 5" and "3 + 7 = 8" respectively. Describe the following statements:

(i) $p \wedge q$ (ii) $\sim p \wedge q$

(iii) $p \wedge \sim q$ (iv) $\sim p \wedge \sim q$

(v) $p \vee q$ (vi) $\sim p \vee q$

(vii) $p \vee \sim q$ (viii) $\sim p \vee \sim q$

5. Find the truth value of the following statements:

(i) $2 + 4 = 6 \wedge 2 \times 3 = 6$

(ii) $2 + 5 = 25 \wedge$ It is falls that $5 + 3 = 8$

(iii) It is falls that $2 + 5 = 8 \wedge 2 \times 5 = 20$

6. Find the truth value of the following statements:

(i) Real part of $(4 + i)^2 = 15 \vee$ roots of $x^2 - 5x + 6$ are 2, 3.

(ii) $\left(2^2\right)^4 = 64$ or $\log_{625} 25 = 2$

(iii) $\cos\left(-135^0\right) = 1 \vee \sec 450^0 = \frac{1}{2}$

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7. If the statements p and q are respectively: "4 + 5 = 9" and "2 + 3 = 5", then write the conditional statements:

(i) $p \Rightarrow q$ (ii) $\sim p \Rightarrow q$

(iii) $p \Rightarrow \sim q$ (iii) $\sim q \Rightarrow \sim p$

8. If the statements p and q are respectively: "3 < 4" and "8 > 3", then write the bi-conditional statements:

(i) $p \Leftrightarrow q$ (ii) $\sim p \Leftrightarrow q$

(iii) $p \Leftrightarrow \sim q$ (iii) $\sim p \Leftrightarrow \sim q$

9. If the truth values of the statements p and q are T and T respectively then write the truth values of:

(i) $p \Rightarrow q$ (ii) $p \Rightarrow \sim q$

(iii) $\sim (p \Leftrightarrow \sim q)$ (iii) $\sim p \Leftrightarrow \sim q$

10. If the truth values of the statements p and q are T and F respectively then write the truth values of:

(i) $p \Rightarrow q$

(ii) $\sim (p \Leftrightarrow \sim q)$

(iii) $\sim (\sim p \Leftrightarrow \sim q)$

11. Construct the truth table of the following statements:

(i) $\sim (p \wedge \sim q)$ (ii) $\sim (\sim p \vee \sim q)$

(iiii) $(p \wedge q) \vee (p \vee q)$ (iv) $(\sim p \vee q) \wedge \sim (p \wedge \sim q)$

(v) $p \wedge (q \wedge r)$ (vi) $(p \vee q) \vee r$

(vii) $p \wedge (q \vee r)$ (viii) $(p \wedge q) \vee r$

(ix) $(p \wedge \sim q) \vee r$ (x) $\sim p \vee (q \wedge \sim r)$

(xi) $(\sim p \wedge \sim q) \vee \sim r$ (xii) $\sim (p \wedge q) \vee \sim r$

12. Construct the truth table of the following statements:

(i) $p \Rightarrow \sim q$ (ii) $\sim p \Rightarrow q$

(iiii) $\sim p \Leftrightarrow q$ (iv) $\sim p \Leftrightarrow \sim q$

(v) $(p \wedge q) \Rightarrow \sim p$ (vi) $(p \wedge q) \Rightarrow (p \vee q)$

(vii) $(p \Rightarrow q) \Rightarrow r$ (viii) $(p \vee q) \Leftrightarrow r$

13. Write the negation of the following statements:

"A function f: IR $\rightarrow$ IR is continuous at a $\in$ IR for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $f(a-\delta,\ a+\delta) \subseteq (f(x)-\varepsilon,\ f(x)+\varepsilon)$ *(A.U. 2013)*

14. Write the negation of the following statements:

"$x \in$ IR is a limit point of a set A $\subseteq$ IR if for each $\varepsilon > 0$, $B^*(x, \varepsilon) \cap A = \phi$, where $B^*(x, \varepsilon) = (x-\varepsilon, x) \cup (x, x+\varepsilon)$. *(A.U. 2013)*

1.24 Tautology: A statement whose truth value is always true (T) is called a **"Tautology".**

A Tautology is also called a **theorem or a logically valid statement pattern.**

1.25 Contradiction: The statement whose truth value is always false (F) is called a **"Contradiction".**

A Contradiction is also called a **fallacy.**

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Note:
(i) The negation of a tautology is a contradiction.
(ii) The negation of a contradiction is a tautology.

Solved Examples

Example 1: Is $(p \wedge q) \Rightarrow (p \vee q)$ a tautology ? *(A.U. 2012)*

Solution: $p \wedge q$ has a true value (T) if both have true values and $p \vee q$ has the truth value false (F) if both are false, Also conditional statement has the truth value false (F) only if antedent (p) is true (T) and consequent (q) is false (F). Thus we have

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \Rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

In the last column all the truth values are T. Hence it is a tautology.

Example 2: Find the negation of $(p \vee q) \wedge (p \Rightarrow q)$ *(A.U. 2012)*

Solution:

p	q	$p \vee q$	$p \Rightarrow q$	$(p \vee q) \wedge (p \Rightarrow q)$	$\sim [(p \vee q) \wedge (p \Rightarrow q)]$
T	T	T	T	T	F
T	F	T	F	F	T
F	T	T	T	T	F
F	F	F	T	F	T

Example 3: Show that $[p \Rightarrow q] \Leftrightarrow [\sim p \vee q]$ is a tautology. *(A.U. 2011)*

Solution:

p	q	$\sim q$	$\sim p \vee q$	$p \Rightarrow q$	$[p \Rightarrow q] \Leftrightarrow [\sim p \vee q]$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

In the last column all the truth values are T. Hence it is a tautology.

Example 4: Prove or disprove $p \vee \sim p \Rightarrow q$ is a tautology. *(A.U. 2011)*

Solution:

p	q	$\sim p$	$p \vee \sim p$	$p \vee \sim p \Rightarrow q$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

In the last column all the truth values are not true. Hence it is not a tautology.

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Example 5: Prove or disprove $p \vee q \Rightarrow p$ is a tautology. *(A.U. 2010)*

Solution: Truth value of $p \vee q \Rightarrow p$

p	q	p ∨ q	p ∨ q ⇒ p
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

In the last column all the truth values are not true. Hence it is not a tautology.

Example 6: Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Solution: Truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	~p	~q	p ∧ q	~p ∧ ~q	(p ∨ q) ∧ (~p ∧ ~q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

In the last column all the truth values are not false. Hence it is a contradiction.

Example 7: Show that $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$ is a tautology.

Solution: Truth value of $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$

p	q	~p	~q	p ∧ q	(p ∧ q) ∧ (~p)	p ∧ ~q	(p ∧ q) ∨ (~p) ∨ (p ∧ ~q)
T	T	F	F	T	T	F	T
T	F	F	T	F	F	T	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	F	T

In the last column all the truth values are true. Hence it is tautology.

Example 8: Show that $[(p \vee q) \vee r] \Leftrightarrow p \vee (q \vee r)$ is a tautology.

Solution: Truth value of $[(p \vee q) \vee r] \Leftrightarrow p \vee (q \vee r)$

p	q	r	p ∨ q	(p ∨ q) ∨ r	q ∨ r	p ∨ (q ∨ r)	[(p ∨ q) ∨ r] ⇔ p ∨ (q ∨ r)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

In the last column all the truth values are true. Hence it is tautology.

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Example 9: Show that $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ is a tautology.

Solution: Truth value of $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T

In the last column all the truth values are true. Hence it is a tautology.

Excercise 1.2

1. Show that $p \vee \sim p$ is a tautology.
2. Show that $p \wedge \sim p$ is a fallacy.
3. Show that $(p \vee q) \Rightarrow p$ is a tautology.
4. Show that $p \Rightarrow (p \vee q)$ is a tautology.
5. Show that $\sim [(p \wedge q) \Rightarrow (p \vee q)]$ is a contradiction.
6. Show that $\sim (p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$ is a tautology.
7. Show that $(p \wedge q) \wedge \sim (p \wedge q)$ is a contradiction.
8. Show that $((\sim p) \wedge (q \wedge r)) \wedge \sim q$ is a tautology.
9. Show that $(p \wedge \sim q) \wedge ((\sim p) \vee q)$ is a fallacy.
10. Show that $((\sim q) \wedge p) \vee (p \vee \sim p)$ is a tautology.
11. Write the contra positive of the following statements:
    (a) "A differentiable function is a continuous function".
    (b) "A function f: IR $\rightarrow$ IR is continuous if f is continuous at each point in IR".

1.26 Logical Equivalence: Two compound statements S_1 (p, q, r,.....) and S_2 (p, q, r,.....) of components p, q, r,..... are called logically equivalent or simply equivalent or equal if they have identical truth values and we write

S_1 (p, q, r,.....) $\equiv S_2$ (p, q, r,.....)

Solved Examples

Example 1: Show that $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

Solution: Truth table of $(p \vee q) \wedge \sim p$

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Truth table of $\sim q \wedge p$

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

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In the last column of both the tables truth values are identical.

Hence $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

Example 2: Show that $(p \Rightarrow q) \equiv (\sim p \vee q)$

Solution: Truth table of $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
T	T	T

Truth table of $(\sim p \vee q)$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	T	T	T

In the last column of both the tables truth values are identical.

Hence $(p \Rightarrow q) \equiv \sim p \vee q$

Example 3: Show that $\sim (p \Leftrightarrow q)$ and $(p \wedge \sim q) \vee (\sim p \wedge q)$ are logically equivalent.

Solution:

Truth table of $\sim (p \Leftrightarrow q)$

p	q	$p \Leftrightarrow q$	$\sim (p \Leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Truth values of $(p \wedge \sim q) \vee (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

In the last column of both the tables truth values are identical.

Hence $\sim (p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

Example 4: Whether the compound statements $(p \wedge q) \vee [\sim p \vee (p \wedge \sim q)]$ and $(\sim p \wedge q) \vee t$ are logically equivalent, where t is a tautology ?

Solution:

Truth table of $(p \wedge q) \vee [\sim p \vee (p \wedge \sim q)]$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \vee (p \wedge \sim q)$	$(p \wedge q) \vee [\sim p \vee (p \wedge \sim q)]$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	F	T	T

Truth table of $(\sim p \wedge q) \vee t$

p	q	$\sim p$	$(\sim p \wedge q)$	t	$(\sim p \wedge q) \vee t$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	T	T

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In the last column of both the tables truth values are identical. Hence given compound statements are equivalent.

Exercise 1.3

1. Prove each of the following:
   (i) $p \wedge (p \vee q) \equiv p$ (ii) $p \vee (p \wedge q) \equiv p$
   (iii) $\sim(p \wedge \sim q) \equiv \sim p \vee q$ (iv) $\sim(\sim p \vee \sim q) \equiv p \wedge a$
   (v) $(p \Rightarrow q) \equiv (\sim p) \vee q$ (vi) $p \Rightarrow q \equiv \sim p \Rightarrow \sim p$
   (vii) $p \Rightarrow q \neq q \Rightarrow p$ (viii) $\sim p \vee q \neq \sim(p \vee q)$
   (ix) $p \Leftrightarrow q) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$ (x) $\sim(p \Rightarrow q) \equiv (\sim p) \Leftrightarrow p \Leftrightarrow \sim q$
   (xi) $(\sim q \wedge p) \wedge (p \wedge \sim p) \neq (p \wedge q) \wedge f$, where f is a fallacy.
2. Show that $\sim(p \vee q) \wedge (p \vee \sim r) \wedge (\sim p \vee \sim q) \neq p \vee r$
3. Show that $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$
4. Prove that $\sim\big(p \Rightarrow (q \wedge \sim r)\big)$ and $(p \Rightarrow q) \wedge (p \Rightarrow r)$ are logically equivalent.

1.27 Algebra of Statements:

I. Idempotent Laws: If p is any statement then

(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

Proof: Truth Table of $p \vee p$ and $p \wedge p$

p	p ∨ p	p ∧ p
T	T	T
F	F	F

$\therefore$ $p \vee p \equiv p$ and $p \wedge p \equiv p$

II. Complement Laws: If p is any statement then

(i) $p \vee \sim p \equiv t$ (ii) $p \wedge \sim p \equiv t$
(iii) $\sim(\sim p) \equiv p$ (iv) $\sim t \equiv f, \sim f \equiv t$

Where t and f are respectively tautology and fallacy in terms of p.

Proof: Truth Table of $p \vee \sim p$, $p \wedge \sim p$, $\sim(\sim p)$, $\sim t$, $\sim f$

p	~p	p ∨ ~p	t	p ∧ ~p	f	~(~p)	~t	~f
T	F	T	T	F	F	T	F	T
F	T	T	T	F	F	F	F	T

$\therefore$ $p \vee \sim p \equiv t$, $p \wedge \sim p \equiv f$, $\sim(\sim p) \equiv p$, $\sim t \equiv f$, $\sim f \equiv t$

III. Identity Laws: If p is any statement, then

(i) $p \wedge t \equiv p$ (ii) $p \wedge f = f$
(iii) $p \vee t \equiv t$ (iv) $p \vee f \equiv p$

Where t and f are respectively tautology and fallacy in terms of p.

Proof: Truth Table of $p \wedge t$, $p \wedge f$, $p \vee t$, $p \vee f$

p	t	f	p ∧ t	p ∧ f	p ∨ t	p ∨ f
T	T	F	T	F	T	T
F	T	F	F	F	T	F

$\therefore$ $p \wedge t \equiv p$, $p \wedge f \equiv f$, $p \vee t \equiv t$, $p \vee f \equiv p$

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IV. Commutative Laws: If p is any statements then

(i) $p \wedge q \equiv q \wedge p$ (ii) $p \vee q \equiv q \vee p$

Proof: Truth Table of $p \wedge q$, $q \wedge p$, $p \vee q$, $q \vee p$

p	q	$p \wedge q$	$q \wedge p$	$p \vee q$	$q \vee p$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

$\therefore$ $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$

V. De-Morgan's Laws: If p is any statements then

(i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Proof: (i) Truth tables of $\sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Truth tables of $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$\therefore$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(ii) Truth tables of $\sim(p \vee q)$

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Truth tables of $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$\therefore$ $\sim(p \vee q) \equiv \sim p \wedge \sim q$

VI. Associative Laws: If p, q, r be any three statements then

(i) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(ii) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Proof: (i) Truth tables of $(p \wedge q) \wedge r$

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
T	F	F	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Truth tables of $p \wedge (q \wedge r)$

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
F	T	T	F	F
T	F	F	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

$\therefore$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(ii) Try yourself.

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VII. Distributive Laws: If p, q, r be any three statements, then

(i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Proof: (i) Truth tables of $p \wedge (q \vee r)$

p	q	r	q ∨ r	p ∧ (q ∨ r)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
F	T	T	T	F
T	F	F	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Truth tables of $(p \wedge q) \vee (p \wedge r)$

p	q	r	p ∧ q	p ∧ r	(p ∧ q) ∨ (p ∧ r)
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
F	T	T	F	F	F
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

$\therefore$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(ii) Try yourself.

Solved Examples

Example 1: By using law of algebra of statements, show that $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

Solution:

$(p \vee q) \wedge \sim p \equiv (\sim p) \wedge (p \vee q)$ (Using Commutative law)

$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$ (Distributive law)

$\equiv f \vee (\sim p \wedge q)$ (Complement law)

$\equiv \sim p \wedge q$ (Identity law)

Example 2: Show that $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Solution: $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv (\sim p \wedge (\sim q \wedge r)) \vee ((q \vee p) \wedge r)$

$\equiv ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)$

$\equiv (\sim p \wedge \sim q) \vee (q \vee p) \wedge r$

$\equiv (\sim (p \vee q) \vee (p \vee q)) \wedge r$

$\equiv t \wedge r$

$= r$

Exercise 1.4

Prove the following logical equivalence using laws of algebra of statements:

(i) $\sim (\sim (\sim p)) \equiv p$

(ii) $\sim (\sim p \sim q) \equiv p \vee \sim q$

(iii) $\sim (p \vee \sim q) \equiv p \wedge q$

(iv) $p \wedge (p \vee q) \equiv p \vee (p \wedge q)$

(v) $p \wedge (\sim p \vee q) \equiv p \wedge q$

(vi) $p \vee (p \wedge q) \equiv p$

(vii) $p \wedge (p \vee q) \equiv p \vee (p \wedge q)$

(viii) $(p \wedge q) \vee \sim p \equiv - p \vee q$

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1.28 Duality: The two compound statements are said to be dual of each other if either one can be obtained from other by interchanging $\wedge$ by $\vee$ and $\vee$ by $\wedge$, tautology t by fallacy f and fallacy f by tautology t.

Example:

(i) $(p \vee q) \wedge r$ and $(p \wedge q) \vee r$ are dual of each other.

(ii) $(p \wedge q) \vee t$ and $(p \vee q) \wedge f$ are dual of each other.

(iii) $\sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim r)$ and $\sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim r)$ are dual of each other.

(iv) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ and $\sim(p \vee q) \equiv \sim p \wedge \sim q$ are dual of each other.

(v) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ are dual of each other.

1.29 Important Result: Let S and S* be the dual compound statement and let p, q, r, ... be finitely many statements that occurs in S and S*, then

$\sim S^*(p, q, r, \ldots\ldots) \equiv S(\sim p, \sim q, \sim r, \ldots\ldots)$

Solved Examples

Example 1: Write the duals of the following compound statements:

(i) $(p \wedge q) \wedge \sim p$ (ii) $(p \vee q) \wedge (p \wedge r)$

(iii) $[(p \vee r) \wedge (p \wedge q)] \vee f$ (iv) $[(p \vee q) \wedge \sim r] \vee (p \wedge t)$

Solution:

(i) The dual of $(p \wedge q) \wedge \sim p$ is $(p \vee q) \vee \sim p$

(ii) The dual of $(p \vee q) \wedge (p \wedge r)$ is $(p \wedge q) \vee (p \vee r)$

(iii) The dual of $[(p \vee r) \wedge (p \wedge q)] \vee f$ is $[(p \wedge r) \vee (p \vee q) \wedge t$

(iv) The dual of $[(p \vee q) \wedge \sim r] \vee (p \wedge t)$ is $[(p \wedge q) \vee \sim r] \wedge (p \vee f)$

Example 2: If $S(p, q, r) \equiv p \vee (q \vee r)$ and $S^*(p, q, r)$ be the dual of $S(p, q, r)$, then verify that $\sim S^*(p, q, r) \equiv S(\sim p, \sim q, \sim r)$

Solution:

$\because \quad S(p, q, r) \equiv p \vee (q \vee r)$

$\therefore \quad S^*(p, q, r) \equiv p \wedge (q \wedge r)$ $\quad$ (Dual of $S(p, q, r)$)

$\Rightarrow \quad \sim S^*(p, q, r) \equiv \sim [p \wedge (q \wedge r)]$

$\equiv \sim p \vee (\sim q \wedge r)$

$\equiv \sim p \vee (\sim q \vee \sim r)$

Also, $S(\sim p, \sim q, \sim r) \equiv \sim p \vee (\sim q \vee \sim r)$

Thus, $\sim S^*(p, q, r) \equiv S(\sim p, \sim q, \sim r)$

Exercise 1.5

1. Write the duals of the following statements:

(i) $(p \vee q) \wedge \sim p$

(ii) $[(p \vee r) \vee (\sim p \vee \sim q)] \wedge r$

(iii) $[(p \wedge \sim q) \vee (\sim p \wedge q)] \vee [p \wedge q \wedge t]$

(iv) $(\sim p \vee f) \wedge [\sim q \wedge (p \vee q) \wedge \sim r]$

(v) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim r)]$

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2. If $S(p, q) \equiv (p \vee \sim q) \wedge p$ and $S^*(p, q)$ be the dual of $S(p, q)$ then verify that $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

1.30 Tautological Implication: A statement p is said to **tautologically imply** a statement q if and only if $p \rightarrow q$ is a tautology. We shall denote this by $p \Rightarrow q$ which is read as "p implies q". Here "$\rightarrow$" stands for "If then" and "$\Rightarrow$" is not a connective nor $p \Rightarrow q$ is a statement.

Note:

(i) $p \Leftrightarrow q$ is similar as $p \Rightarrow$ and $p \Rightarrow q$ state that p and q equivalent or that $p \Leftrightarrow$ is a tautology.

(ii) $p \Rightarrow q$ then it is a tautology and this guarantees that q has the truth value T wherever p has the truth value T.

(iii) If a compound statement is equivalent to a tautology, then it must be a tautology.

(iv) If a compound statement is implied by a tautology, then it is a tautology.

1.31 Rules of Implications:

1.	$p \wedge q \Rightarrow p$
2.	$p \wedge q \Rightarrow q$
3.	$p \Rightarrow p \vee q$
4.	$\sim p \Rightarrow p$
5.	$q \Rightarrow p \rightarrow q$
6.	$\sim (p \rightarrow q) \Rightarrow \sim q$
7.	$\sim (p \rightarrow q) \Rightarrow p$
8.	$p \wedge (p \rightarrow q) \Rightarrow q$
9.	$\sim q \wedge (p \rightarrow q) \Rightarrow \sim p$
10.	$\sim p \wedge (p \vee q) \Rightarrow q$
11.	$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$
12.	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) = r$

1.32 Implication is transitive: Consider that $p \Rightarrow q$ and $q \Rightarrow r$, then $p \rightarrow q$ and $q \rightarrow r$ are tautologies. Hence $(p \rightarrow q) \wedge (q \rightarrow r)$ is also a tautology. But from implication rule (II), $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$. Hence $q \rightarrow r$ is a tautology.

Note:

(i) If $p \Rightarrow q$ and $p \Rightarrow r$ then $p \Rightarrow (q \wedge r)$

(ii) If $H_1, H_2, \ldots\ldots, H_m$ and p imply q, then $H_1, H_2, \ldots\ldots, H_m$ imply $p \rightarrow q$.

1.33 The Statement Function and Quantifiers:

Statement Function: Let H be a given set. A statement function on H is an expression $H(x)$ which has property that $H(a)$ is true for each $a \in H$ is substituted for the variable x. The set H is called domain of $H(x)$ and the set T_H of all the elements of H for which $H(a)$ is true is called the truth set of $H(x)$ i.e.

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$T_H = \{x : x \in H, H(x) \text{ is true}\}$

For example:

Let $H(x) = x + 5 > 1$.

Its true set is

$N = \{x : x \in N, x + 5 > 1\}$

Thus $H(x)$ is true for every element in N.

Let $M(x)$ be "x is a man" and $H(x)$ be "x is a mortal", Then compound statement functions are:

$M(x) \wedge H(x)$, $M(x) \Rightarrow H(x)$, $\sim H(x)$, $M(x) \vee H(x)$ etc.

Statement Functions of two variables:

Let $G(x, y)$: x is taller than y.

If both x and y are substituted by name of objects, we get a statement. If m represents Monu and represents Sonu, then we have

G (m, s): Monu is taller than Sonu.

and G (s, m): Sonu is taller than Monu.

It is possible to form statement function of two variables by using statement function of one variable.

For example: Let $M(x)$: x is a man and M(y): y is a mortal.

Then $M(x) \wedge M(y)$: x is a man and y is a mortal.

> ✍ **Note:** However, it is not possible to write every statement function of two variables using statement function of one variable.

Universal Quantifiers: Let $H(x)$ be a statement function defined on a set A. Consider the expression [$\forall x \in H$, $H(x)$ is true], then the symbol $\forall$ (for all is called the **universal quantifier.** This symbol be placed before the statement function to which this phrase is applied. This symbol is quantification symbol.

For example: Let $M(x)$: x is a man and $H(x)$: x is a mortal. If we introduce a symbol to denote the phrase "for all x", then it would be possible to symbolize "for all x, if x is man, then x is a mortal as $(\forall x)(M(x) \rightarrow H(x))$.

Existential Quantifiers: Let $H(x)$ be a statement function defined on a set A. Consider the expression [$\exists x \in A$, $H(x)$ is true], then the symbol $\exists$ (there exist) is called **existential quantifier.**

($\exists x$) symbolized expression such that "there exist an x such that".

For example: If $M(x)$: x is a man and $C(x)$: x is clever then the statement "Some men are clever can be expressed as $(\exists x)(M(x) \wedge C(x))$.

Multiple Quantifiers: If a statement have more than one variable then we can quantify it more than once.

For example: G (m, s): m is taller than s.

> ✍ **Note:** The particular variable appearing in the statements involving a quantifier is not important because the statements remains unchanged if x is replaced by y-throughout. Then Some apples are red. and $(\exists y)(M(y) \rightarrow H(y))$ are equivalent.

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Solved Examples

Example 1: If $H(x)$ is a formula in x, then translate the following:

(i) $A(x)\,H(x)$ (ii) $(\forall x)[\sim H(x)]$

(iii) $\exists(x)\,H(x)$ (iv) $(\exists x)[\sim H(x)]$

Solution:

(i) Every x has the property $H(x)$.

(ii) No x has property $H(x)$.

(iii) There exist x which has property $H(x)$.

(iv) There is some x which do not have property $H(x)$.

Example 2: Translate the following statements, involving quantifiers into formula:

(i) All apples are red. (ii) No apples are red.

(iii) Some apples are red. (iv) Some apples are not red.

Solution: Let x be an individual variable and $A(x)$: x is an apple and $R(x)$: x is red. Then

(i) $(\forall x)[\forall(x) \to R(x)]$ (ii) $(\forall x) \sim [\forall(x) \to R(x)]$

(iii) $(\exists x)[\forall(x) \wedge R(x)]$ (iv) $(\exists x)[\forall(x) \wedge \sim R(x)]$

Exercise 1.6

1. Show the following implications:

   (i) $p \Rightarrow (q \to p)$ (ii) $(p \wedge q) \Rightarrow (p \to q)$

2. Show the following implications without constructing the truth tables:

   (i) $p \to q \Rightarrow (p \wedge q)$

   (ii) $(p \to q) \to q \Rightarrow p \vee q$

3. Translate the statements involving quantifiers into formulae:

   (i) All rational are real. (ii) No rational are real.

   (iii) Some rational are real. (iv) Some rational are not real.

~STS